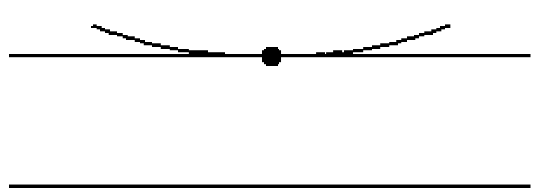
Ball and Saddle Geometries

2,300 years ago, Euclid wrote a treatise on geometry called The Elements. Basing it all on ten postulates, he codified geometry into a rigorous branch of mathematics, the first such branch and for most of history since the only such branch.

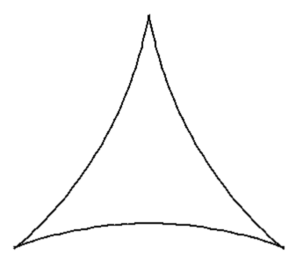
Early on, long before the fall of the Roman Empire though centuries after Euclid, one small fly in the ointment appeared. One of the self-evident postulates no longer seemed self-evident. This is the parallel postulate.

In modern form, it says that given a line and one point not on that line, exactly one other line can be drawn through that point parallel to the given line. The problem with this postulate is that it asserts what happens or doesn't happen arbitrarily far away, namely that the two lines never meet. Such an assertion cannot be called self-evident.



Mathematicians tried deducing it as a theorem from the other nine postulates but failed. In the 1800s, the mathematicians Bolyai and Lobachevsky, working separately, tried a contrary assumption, shown in the diagram above. It shows two parallel lines through the point, but one is drawn curved so we can see the two parallels as distinct. Bolyai and Lobachevsky found this assumption led to crazy but consistent results. Both had the insight to realize these results were not wrong, they were just the theorems of a new geometry, a saddle geometry.

If you take a surface with negative curvature like a horse's saddle,[1] Euclid's postulate is false on that surface and Bolyai and Lobachevsky's is true. Flattening an equilateral triangle from a saddle onto a plane gives the figure below, the sides being straight lines within the saddle surface, where a straight line is whatever path is shortest in connecting two points.



While it would be nice to have an example of a two-dimensional surface with *uniform* negative curvature (the same curvature everywhere) and there is such a surface, it takes four spatial dimensions to realize this two-dimensional surface.[2] For a surface of constant *positive* curvature three dimensions is enough and we have the surface of a sphere.

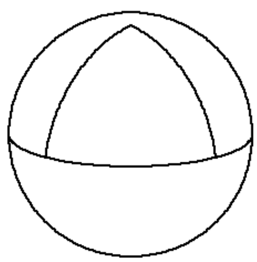
How about assuming zero parallel lines through the given point? That did not work. It was easy to derive contradictions. Then Bernhard Riemann came along. Riemann changed another postulate of Euclid, the one saying straight lines could be infinitely extended. Riemann argued that instead of infinite extension we should assume unbounded extension. A great circle of a

[1] In one direction the saddle curves downward while in the other direction it curves upward. This is negative curvature. Compare it to a sphere.

[2] Just as it takes three spatial dimensions to realize a knot, which is a one-dimensional object. Just one extra dimension is not always enough.

sphere is finite, yet it is unbounded. With this additional change, Riemann developed a second self-consistent non-Euclidean geometry that applies to positively curved surfaces such as a sphere.

Euclid’s geometry is about flat space. In his geometry the sum of the three angles of any triangle is 180° and the Pythagorean Theorem holds. For both other geometries the angles never sum to 180°, the deviation depending on the triangle’s area. The Pythagorean Theorem never holds.



Ball geometry is easier to picture than saddle geometry, so try picturing a triangle containing one octant of the earth (diagram above), that is, one fourth of the equator plus lines of longitude connecting each of the two endpoints of that equator arc to the north pole. Each angle of this triangle is 90° and the sum of the angles is 270°. It is an equilateral triangle with three right angles.

Straight lines on a sphere are its great circles and all great circles intersect (in two places[3]) so there are no parallel lines at all.[4] All straight lines are unbounded yet finite.

[3] It is possible to construct a ball geometry in which all pairs of straight lines intersect at just one point, but these geometries are weird.

[4] If you are wondering about lines of latitude, remember that except for the equator they are not great circles and so are not straight lines, the shortest paths between points. Airline routes are straight lines.