Calculus of Variations

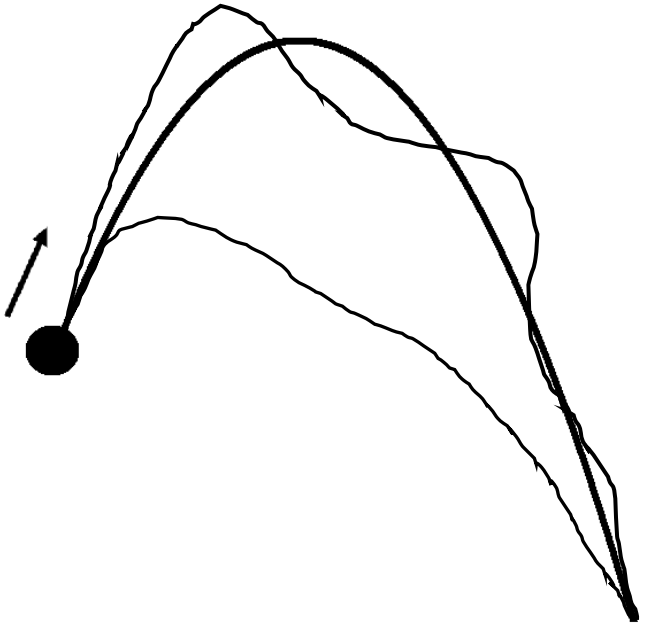
I hate using the word “calculus” because it scares people but try reading just a little further.

Suppose you must draw a line that is minimal or maximal in some way. Perhaps it must be the shortest possible line connecting two points (a straight line), or the line of fixed length surrounding the largest possible area (a circle), or the shortest line on the surface of a horse’s saddle connecting a certain two points on the saddle (that one’s tougher). Finding the best line in each case is in the purview of the Calculus of Variations.

Other examples are in order. A grandfather clock is tall for several reasons. One is that a pendulum does not take the same amount of time for swings of different sizes. By making it tall, the angle of the swing is kept small and all swings do take pretty much the same time, but not exactly. If you designed a clock so the time of the swing was truly independent of the size of the swing, you would end up using cycloids, per the previous vignette. How do we know? Calculus of variations.

Finally, you have probably heard of the Principle of Least Action. This phrase gets abused, being applied in fields where it is irrelevant. It is a principle of physics formulated and solved by the calculus of variations.

Throw a ball into the air at any angle. The ball follows an arc due to gravity. Why that exact arc? What is special about that arc? That is the one and only path minimizing the Action, a numeric quantity defined in physics. Computing the Action for any other path such as the others in the diagram produces a larger value.



What do I mean by “path”? The meaning is stringent. A path is more than just the locus of points followed by the ball. It includes precisely when the ball is at each point. A second ball moving on the same locus of points but not at the same point at the same time is traveling a different path. With this understanding the Principle of Least Action gives us the exact path followed by the ball.

For a different problem, say electromagnetic instead of gravitational, we compute the electromagnetic action but the Principle still holds. We get the different path an electrically charged ball follows.

Calculus of Variations is not ordinary calculus, although it does make use of the tools provided by ordinary calculus. Its impact on physics is far greater than I have been able to describe here.